Minimal sparsity for scalable moment-SOS relaxations of the AC-OPF problem

Workshop of the RTE Chair at CentraleSupélec

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May the 25th, 2023

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Outline of the presentation

1. Background on the moment hierarchy for POPs
2. The AC-OPF problem: POP formulation and scalability issue
3. Addressing large scale instances ?
4. Conclusion and perspective

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Polynomial Optimization Problem

$$\rho = \min_{x \in \mathcal{X}} f(x) \qquad \text{(POP)}$$

- $\mathcal{X} = \{x \in \mathbb{R}^n \mid g_j(x) \geq 0\,, \ \ \forall j \in [\![1, m]\!]\,\}$
- f and all g_j are polynomial functions
- we assume that (POP) has a solution
  (e.g. $\mathcal{X}$ is nonempty and compact)

Example (a nonconvex QCQP)

$$\begin{aligned} \rho = \min_{x \in \mathbb{R}^2} \quad & x_1 \\ s.t. \quad & 2x_1 - x_2 + 1 \geq 0 \\ & 2x_1 + x_2 + 1 \geq 0 \\ & x_1^2 + x_2^2 = 1 \end{aligned}$$

POP as a moment problem

$$\rho = \min_{x \in \mathcal{X}} f(x) = \inf_{\mu \in \mathcal{M}(\mathcal{X})} \int_{\mathbb{R}^n} f(x)\mu(dx)$$

POP as a moment problem

$$\rho = \min_{x \in \mathcal{X}} f(x) = \inf_{\mu \in \mathcal{M}(\mathcal{X})} \int_{\mathbb{R}^n} f(x)\mu(dx)$$
$$= \inf_{\mu \in \mathcal{M}(\mathcal{X})} \sum_{\alpha \in \mathrm{supp}(f)} f_\alpha \int_{\mathbb{R}^n} x^\alpha \mu(dx)$$

POP as a moment problem

$$
\begin{aligned}
\rho = \min_{x \in \mathcal{X}} f(x) &= \inf_{\mu \in \mathcal{M}(\mathcal{X})} \int_{\mathbb{R}^n} f(x)\mu(dx) \\
&= \inf_{\mu \in \mathcal{M}(\mathcal{X})} \sum_{\alpha \in \mathrm{supp}(f)} f_\alpha \int_{\mathbb{R}^n} x^\alpha \mu(dx) \\
&= \inf \left\{ \sum_{\alpha \in \mathrm{supp}(f)} f_\alpha y_\alpha \,\middle|\, \text{“}y \text{ has a representing measure on } \mathcal{X}\text{”} \right\}
\end{aligned}
$$

POP as a moment problem

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$$= \inf \left\{ \sum_{\alpha\in\mathrm{supp}(f)} f_\alpha y_\alpha \,\middle|\, \text{“}y \text{ has a representing measure on } \mathcal{X}\text{”} \right\}$$

Proposition (necessary condition)

If $y \in \mathbb{R}^{\mathbb{N}^n_{2d}}$ is the sequence of moments (up to order $2d$) of a measure supported by the set $\mathcal{X}$, then

- $M_d(y) \succeq 0$ *(moment matrix)*
- $M_{d-d_j}(g_j y) \succeq 0\,,\ \forall j \in [\![1, m]\!]$ *(localizing matrices)*

Moment matrices ?

$$M_d(y) = \left(y_{\alpha+\beta}\right)_{\alpha\in\mathbb{N}^n_d,\beta\in\mathbb{N}^n_d}$$

$$M_{d-d_j}(g_j y) = \left(\sum_{\gamma\in\mathrm{supp}(g_j)} g_{j,\gamma} y_{\alpha+\beta+\gamma}\right)_{\alpha\in\mathbb{N}^n_{d-d_j},\beta\in\mathbb{N}^n_{d-d_j}} \qquad \left(d_j = \left\lceil\frac{\deg(g_j)}{2}\right\rceil\right)$$

Example

For $n = 2$ and $d = 1$, $M_d(y) \succeq 0$ writes as

$$\begin{pmatrix} y_{00} & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{pmatrix} \succeq 0$$

The truncated moment hierarchy

$$
\begin{aligned}
\rho_d^{\mathrm{MOM}} = \inf_{y \in \mathbb{R}^{\mathbb{N}^n_{2d}}} \quad & \sum_{\alpha \in \mathrm{supp}(f)} f_\alpha y_\alpha && (\mathrm{MOM}_d) \\
\text{s.t.} \quad & M_d(y) \succeq 0 \\
& M_{d-d_j}(g_j y) \succeq 0 \,, \quad \forall j \in [\![1, m]\!] \\
& y_{0,\dots,0} = 1
\end{aligned}
$$

The truncated moment hierarchy

$$\rho_d^{\text{MOM}} = \inf_{y \in \mathbb{R}^{\mathbb{N}^n_{2d}}} \sum_{\alpha \in \text{supp}(f)} f_\alpha y_\alpha \qquad (\text{MOM}_d)$$

$$\text{s.t.} \quad M_d(y) \succeq 0$$

$$M_{d-d_j}(g_j y) \succeq 0\ , \ \ \forall j \in [\![1, m]\!]$$

$$y_{0,\dots,0} = 1$$

Theorem (Lasserre [2001])

If the set $\mathcal{X}$ is compact and satisfies an Archimedeaness property, then the monotonous non-decreasing sequence of values $\{\rho_d^{MOM}\}_{d\in\mathbb{N}}$ of (MOM_d) converges to the value ρ of (POP).

NB: Archimedeaness can be enforced
by adding a redundant ball constraint to $\mathcal{X}$

Moment relaxations are semidefinite programs

Example (nonconvex QCQP continued)

$$
\begin{aligned}
\rho_1^{\mathrm{MOM}} = \min_{y \in \mathbb{R}^6} \quad & y_{10} \\
s.t. \quad & \begin{pmatrix} y_{00} & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{pmatrix} \succeq 0 \\
& 2y_{10} - y_{01} + 1 \geq 0 \\
& 2y_{10} + y_{01} + 1 \geq 0 \\
& y_{20} + y_{02} - 1 = 0 \\
& y_{00} = 1
\end{aligned}
$$

Moment relaxations are semidefinite programs

Example (nonconvex QCQP continued)

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& y_{20} + y_{02} - 1 = 0 \\
& y_{00} = 1
\end{aligned}
$$

bound	ρ_1^{MOM}	ρ_2^{MOM}	$\bar{\rho}$ (NLP)
value	-0.50	0.00	0.00

Key takeaway

Convergence of the Moment-SOS hierarchy of semidefinite programs

$$
\begin{array}{ccc}
 & \rho = \min_{x \in \mathcal{X}} f(x) & \\
\nearrow & & \nwarrow \\
\vdots & & \vdots \\
\rho_{d+1}^{\mathrm{MOM}} & \geq & \rho_{d+1}^{\mathrm{SOS}} \\
\geq & & \geq \\
\rho_{d}^{\mathrm{MOM}} & \geq & \rho_{d}^{\mathrm{SOS}}
\end{array}
$$

$$(d \geq \max_{j \in [\![0,m]\!]} d_j)$$

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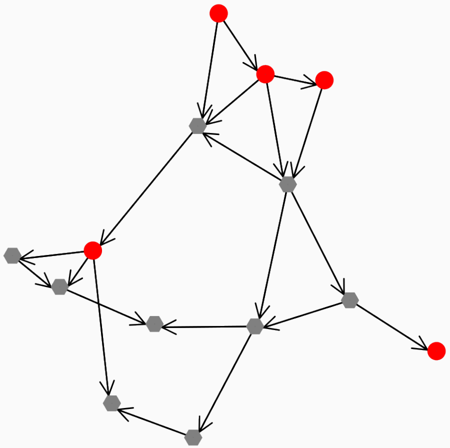
Power grid data

PGLib's case 14 IEEE



Notations for the AC-OPF problem

$$\min_{\substack{v\in\mathbb{C}^{|\mathcal{N}|}\\ s\in\mathbb{C}^{|\mathcal{G}|}}} \quad \sum_{g\in\mathcal{G}} C_{2,g}\Re(s_g)^2 + C_{1,g}\Re(s_g) + C_{0,g} \tag{AC-OPF}$$

$$\text{s.t.} \quad \angle v_i = 0\ ,\ \ \forall i \in \mathcal{N}_r$$

$$\underline{S}_g \le s_g \le \overline{S}_g\ ,\ \ \forall g \in \mathcal{G}$$

$$\underline{V}_i \le |v_i| \le \overline{V}_i\ ,\ \ \forall i \in \mathcal{N}$$

Notations for the AC-OPF problem

$$\min_{\substack{v \in \mathbb{C}^{|\mathcal{N}|} \\ s \in \mathbb{C}^{|\mathcal{G}|} \\ s^{\ell} \in \mathbb{C}^{2|\mathcal{E}|}}} \quad \sum_{g \in \mathcal{G}} C_{2,g} \Re(s_g)^2 + C_{1,g} \Re(s_g) + C_{0,g} \tag{AC-OPF}$$

$$\text{s.t.} \quad \angle v_i = 0 \,, \ \forall i \in \mathcal{N}_r$$

$$\underline{S}_g \leq s_g \leq \overline{S}_g \,, \ \forall g \in \mathcal{G}$$

$$\underline{V}_i \leq |v_i| \leq \overline{V}_i \,, \ \forall i \in \mathcal{N}$$

$$\sum_{g \in \mathcal{G}(i)} s_g - L_i - \left(Y_i^s\right)^* |v_i|^2 = \sum_{j \in \mathcal{N}(i)} s^{\ell}_{i,j} \,, \ \forall i \in \mathcal{N}$$

$$s^{\ell}_{i,j} = \left(Y_{i,j} + Y^c_{i,j}\right)^* \frac{|v_i|^2}{|T_{i,j}|^2} - Y^*_{i,j} \frac{v_i v_j^*}{T_{i,j}} \,, \ \forall (i,j) \in \mathcal{E}$$

$$s^{\ell}_{j,i} = \left(Y_{i,j} + Y^c_{j,i}\right)^* |v_j|^2 - Y^*_{i,j} \frac{v_i^* v_j}{T^*_{i,j}} \,, \ \forall (i,j) \in \mathcal{E}$$

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$$\sum_{g \in \mathcal{G}(i)} s_g - L_i - \left(Y_i^s\right)^* |v_i|^2 = \sum_{j \in \mathcal{N}(i)} s_{i,j}^\ell \ , \ \ \forall i \in \mathcal{N}$$

$$s_{i,j}^\ell = \left(Y_{i,j} + Y_{i,j}^c\right)^* \frac{|v_i|^2}{|T_{i,j}|^2} - Y_{i,j}^* \frac{v_i v_j^*}{T_{i,j}} \ , \ \ \forall (i,j) \in \mathcal{E}$$

$$s_{j,i}^\ell = \left(Y_{i,j} + Y_{j,i}^c\right)^* |v_j|^2 - Y_{i,j}^* \frac{v_i^* v_j}{T_{i,j}^*} \ , \ \ \forall (i,j) \in \mathcal{E}$$

$$|s_{i,j}^\ell| \leq \overline{S}_{i,j} \ , \ \ \forall j \in \mathcal{N}(i) \ , \ \ \forall i \in \mathcal{N}$$

$$\underline{\Theta}_{i,j} \leq \angle v_i v_j^* \leq \overline{\Theta}_{i,j} \ , \ \ \forall (i,j) \in \mathcal{E}$$

Polynomial optimization for AC-OPF

$$v_i = a_i + \mathbb{i} b_i \ , \quad \forall i \in [\![1, n]\!]$$

Example (complex line power)

$$s^{\ell}_{i,j} = Z_{i,j}|v_i|^2 + Z'_{i,j} v_i v_j^*$$

$$\iff$$

$$\begin{cases} \Re(s^{\ell}_{i,j}) = \Re(Z_{i,j})(a_i^2 + b_i^2) + \Re(Z'_{i,j})(a_i a_j + b_i b_j) - \Im(Z'_{i,j})(a_j b_i - a_i b_j) \\ \Im(s^{\ell}_{i,j}) = \Im(Z_{i,j})(a_i^2 + b_i^2) + \Im(Z'_{i,j})(a_i a_j + b_i b_j) + \Re(Z'_{i,j})(a_j b_i - a_i b_j) \end{cases}$$

The AC-OPF problem can be written in form (POP)!

Scalability issue

- AC-OPF IEEE case 57 (no line/angle limits) $\rightarrow$ POP

	(POP)
variables	128
eq. constraints	115
ineq. constraints	128

Scalability issue

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	(POP)
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- POP $\rightarrow$ moment relaxation

	$d = 1$	$d = 2$
size(y)	8.385	12.082.785

ρ_2^{MOM} for PGLib's case 57 IEEE is intractable!
(with LAAS computers and current SDP solvers)

Outline of the presentation

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3. Addressing large scale instances ?

Correlative sparsity

Minimal sparsity

Exploiting sparsity for POPs

$$\min_{x \in \mathbb{R}^3} x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3$$

Exploit absence of $x_1 x_3$ product ?

Exploiting sparsity for POPs

$$\min_{x \in \mathbb{R}^3} x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3$$

Exploit absence of $x_1 x_3$ product ? Set $\mathcal{I}_1 = \{1, 2\}$, $\mathcal{I}_2 = \{2, 3\}$

$$M_1(y) = \begin{pmatrix} y_{000} & y_{100} & y_{010} & y_{001} \\ y_{100} & y_{200} & y_{110} & y_{101} \\ y_{010} & y_{110} & y_{020} & y_{011} \\ y_{001} & y_{101} & y_{011} & y_{002} \end{pmatrix} \succeq 0 \quad \text{vs} \quad \begin{cases} M_1(y|\mathcal{I}_1) \succeq 0 \\ M_1(y|\mathcal{I}_2) \succeq 0 \end{cases}$$

Exploiting sparsity for POPs

$$\min_{x \in \mathbb{R}^3} x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3$$

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Reduce moment variables (MOM_d) / matrices size (SOS_d)

A sparse moment hierarchy

$$
\begin{aligned}
\rho_d^{\text{CS-MOM}} = \inf_y \quad & \sum_{\alpha \in \text{supp}(f)} f_\alpha y_\alpha && (\text{CS-MOM}_d) \\
\text{s.t.} \quad & M_d(y|\mathcal{I}_k) \succeq 0 \ , \ \ \forall k \in [\![1, p]\!] \\
& M_{d-d_j}(g_j y|\mathcal{I}_k) \succeq 0 \ , \ \ \forall j \in [\![1, m]\!] \ , \ \ \forall k \in [\![1, p]\!] \\
& y_{0,\ldots,0} = 1
\end{aligned}
$$

A sparse moment hierarchy

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& y_{0,\ldots,0} = 1
\end{aligned}
$$

Theorem (Lasserre [2006])

If the set $\mathcal{X}$ is compact and satisfies an Archimedeaness property, and if the variable set $\mathcal{I} = \{\mathcal{I}_k\}_{k \in [\![1,p]\!]}$ satisfies the running intersection property (RIP), then the monotonous non-decreasing sequence of values $\{\rho_d^{CS\text{-}MOM}\}_{d \in \mathbb{N}}$ of (CS-MOM$_d$) converges to the value ρ of (POP).

NB: the maximum cliques of a chordal graph satisfy the RIP

Application to AC-OPF

IEEE case 57 after chordal extension + cliques: $\begin{cases} |\mathcal{I}| = 52 \\ \max_{k \in [\![1,p]\!]} |\mathcal{I}_k| = 26 \end{cases}$

- POP $\rightarrow$ sparse moment relaxation

	$d = 1$	$d = 2$
size(y)	1.950	122.286

- numerical result (IEEE case 57 perturbed):

	value	gap to $\bar{\rho}$ (%)	time (s)
$\bar{\rho}$	2433.89	-	4.18
$\rho_2^{\text{CS-MOM}}$	2433.89	0.00	19,666.82
$\rho_1^{\text{CS-MOM}}$	2359.58	3.05	0.75

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3. Addressing large scale instances ?

Correlative sparsity

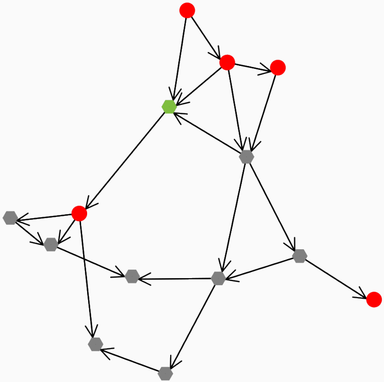
Minimal sparsity

Sparsity sets and scalability

- Interior point SDP solvers scale roughly in $O(N^3)$ with $N = \binom{m+d}{d}$ and $m = \max_{k \in [\![1,p]\!]} |\mathcal{I}_k|$
- It is difficult to control the cardinalities of the sets $\{\mathcal{I}_k\}_{k \in [\![1,p]\!]}$ obtained by chordal extension + cliques

We introduce **minimal sparsity**
designed to reduce the cardinalities of the sets $\{\mathcal{I}_k\}_{k \in [\![1,p]\!]}$
in AC-OPF

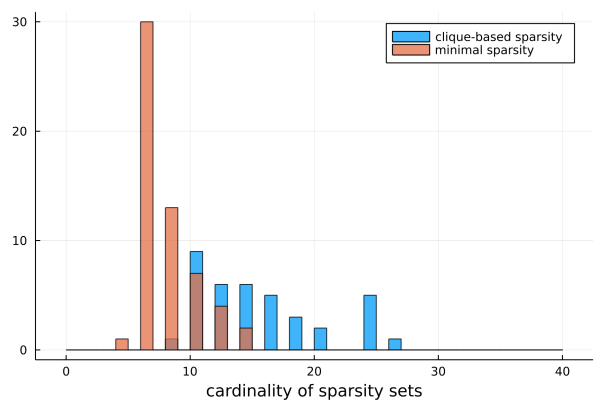
Minimal sparsity based on power flow equations



$$\{x_n\}_{n \in \mathcal{I}^{\mathrm{m}}_{\#i}} = \{\Re(v_i), \Im(v_i)\} \bigcup_{j \in \mathcal{N}(i)} \{\Re(v_j), \Im(v_j)\} \bigcup_{g \in \mathcal{G}(i)} \{\Re(s_g), \Im(s_g)\}$$

More but smaller sparsity sets

PGLib's case 57 IEEE



Second-order moment relaxations via minimal sparsity

- PGLib's case 57 IEEE

	size(y)	time (s)	value
ρ_2	12,082,785	$*$	$*$
$\rho_2^{\text{CS-MOM}}$	122,286	19,666	2433.89
$\rho_2^{\text{MS-MOM}}$	23,526	45	2433.89

Second-order moment relaxations via minimal sparsity

- PGLib's case 57 IEEE

	size(y)	time (s)	value
ρ_2	12,082,785	$*$	$*$
$\rho_2^{\text{CS-MOM}}$	122,286	19,666	2433.89
$\rho_2^{\text{MS-MOM}}$	23,526	45	2433.89

- Large scale instances ?

cases	gap (%)	time (s)
2868 RTE SAD	0.39	6,981
6468 RTE TYP	0.27	12,723
6470 RTE TYP	0.74	15,662

Outline of the presentation

Further improvements for very large scale problems

AC-OPF formulates as a POP but...
typical instances in France have over 6000 nodes!

- Sparsity can help to address very large problems
- Minimal sparsity looks promizing
  to compute second-order relaxations of large instances
- We obtain very large SDPs whose numerical stability
  needs to be improved (future work)

References I

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