Minimal sparsity for scalable moment-SOS relaxations of the AC-OPF problem

Workshop of the RTE Chair at CentraleSupélec

Adrien Le Franc
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LAAS CNRS, Toulouse, France
Outline of the presentation

1. Background on the moment hierarchy for POPs

2. The AC-OPF problem: POP formulation and scalability issue

3. Addressing large scale instances?

4. Conclusion and perspective
Outline of the presentation

1. Background on the moment hierarchy for POPs

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4. Conclusion and perspective
Polynomial Optimization Problem

\[ \rho = \min_{x \in \mathcal{X}} f(x) \]  \hspace{1cm} (POP)

- \( \mathcal{X} = \{ x \in \mathbb{R}^n \mid g_j(x) \geq 0, \ \forall j \in [1, m] \} \)
- \( f \) and all \( g_j \) are polynomial functions
- we assume that (POP) has a solution (e.g. \( \mathcal{X} \) is nonempty and compact)

Example (a nonconvex QCQP)

\[ \rho = \min_{x \in \mathbb{R}^2} x_1 \]
\[ s.t. \]
\[ 2x_1 - x_2 + 1 \geq 0 \]
\[ 2x_1 + x_2 + 1 \geq 0 \]
\[ x_1^2 + x_2^2 = 1 \]
POP as a moment problem

\[
\rho = \min_{x \in \mathcal{X}} f(x) = \inf_{\mu \in \mathcal{M}(\mathcal{X})} \int_{\mathbb{R}^n} f(x) \mu(dx)
\]
\[
\rho = \min_{x \in \mathcal{X}} f(x) = \inf_{\mu \in \mathcal{M}(\mathcal{X})} \int_{\mathbb{R}^n} f(x) \mu(dx) \\
= \inf_{\mu \in \mathcal{M}(\mathcal{X})} \sum_{\alpha \in \text{supp}(f)} f_\alpha \int_{\mathbb{R}^n} x^\alpha \mu(dx)
\]
\[ \rho = \min_{x \in \mathcal{X}} f(x) = \inf_{\mu \in \mathcal{M}(\mathcal{X})} \int_{\mathbb{R}^n} f(x) \mu(dx) \]

\[ = \inf_{\mu \in \mathcal{M}(\mathcal{X})} \sum_{\alpha \in \text{supp}(f)} f_\alpha \int_{\mathbb{R}^n} x^\alpha \mu(dx) \]

\[ = \inf \left\{ \sum_{\alpha \in \text{supp}(f)} f_\alpha y_\alpha \mid "y \text{ has a representing measure on } \mathcal{X}" \right\} \]
POP as a moment problem

\[ \rho = \min_{x \in \mathcal{X}} f(x) = \inf_{\mu \in \mathcal{M}(\mathcal{X})} \int_{\mathbb{R}^n} f(x) \mu(dx) \]

\[ = \inf_{\mu \in \mathcal{M}(\mathcal{X})} \sum_{\alpha \in \text{supp}(f)} f_{\alpha} \int_{\mathbb{R}^n} x^\alpha \mu(dx) \]

\[ = \inf \left\{ \sum_{\alpha \in \text{supp}(f)} f_{\alpha} y_{\alpha} \mid "y \text{ has a representing measure on } \mathcal{X}" \right\} \]

**Proposition (necessary condition)**

*If \( y \in \mathbb{R}^{N_2d} \) is the sequence of moments (up to order 2d) of a measure supported by the set \( \mathcal{X} \), then*

- \( M_d(y) \succeq 0 \) (moment matrix)
- \( M_{d-d_j}(g_jy) \succeq 0 \), \( \forall j \in [1, m] \) (localizing matrices)
\[ M_d(y) = (y_{\alpha+\beta})_{\alpha \in \mathbb{N}_d^n, \beta \in \mathbb{N}_d^n} \]

\[ M_{d-d_j}(g_j y) = \left( \sum_{\gamma \in \text{supp}(g_j)} g_j,\gamma y_{\alpha+\beta+\gamma} \right)_{\alpha \in \mathbb{N}^{n-d_j}, \beta \in \mathbb{N}^{n-d_j}} \quad (d_j = \lceil \text{deg}(g_j) / 2 \rceil) \]

**Example**

For \( n = 2 \) and \( d = 1 \), \( M_d(y) \succeq 0 \) writes as

\[
\begin{pmatrix}
y_{00} & y_{10} & y_{01} \\
y_{10} & y_{20} & y_{11} \\
y_{01} & y_{11} & y_{02}
\end{pmatrix} \succeq 0
\]
The truncated moment hierarchy

\[ \rho_d^{\text{MOM}} = \inf_{y \in \mathbb{R}^{N_{2d}}} \sum_{\alpha \in \text{supp}(f)} f_\alpha y_\alpha \]

s.t.  
\[ M_d(y) \succeq 0 \]
\[ M_{d-j}(g_j y) \succeq 0, \quad \forall j \in [1, m] \]
\[ y_0, \ldots, 0 = 1 \]
The truncated moment hierarchy

\[ \rho_d^{\text{MOM}} = \inf_{y \in \mathbb{R}^{N \times 2d}} \sum_{\alpha \in \text{supp}(f)} f_\alpha y_\alpha \]

\[ \text{s.t.} \quad M_d(y) \succeq 0 \]
\[ M_{d-d_j}(g_j y) \succeq 0, \quad \forall j \in [1, m] \]
\[ y_0, \ldots, 0 = 1 \]

Theorem (Lasserre [2001])

If the set \( \mathcal{X} \) is compact and satisfies an Archimedeaness property, then the monotonous non-decreasing sequence of values \( \{\rho_d^{\text{MOM}}\}_{d \in \mathbb{N}} \) of \( (\text{MOM}_d) \) converges to the value \( \rho \) of \( (\text{POP}) \).

NB: Archimedeaness can be enforced by adding a redundant ball constraint to \( \mathcal{X} \)
Moment relaxations are semidefinite programs

Example (nonconvex QCQP continued)

\[ \rho_{1}^{MOM} = \min_{y \in \mathbb{R}^6} y_{10} \]

s.t. \[
\begin{pmatrix}
  y_{00} & y_{10} & y_{01} \\
  y_{10} & y_{20} & y_{11} \\
  y_{01} & y_{11} & y_{02}
\end{pmatrix} \succeq 0
\]

\[ 2y_{10} - y_{01} + 1 \geq 0 \]
\[ 2y_{10} + y_{01} + 1 \geq 0 \]
\[ y_{20} + y_{02} - 1 = 0 \]
\[ y_{00} = 1 \]
Moment relaxations are semidefinite programs

Example (nonconvex QCQP continued)

\[ \rho_{1}^{\text{MOM}} = \min_{y \in \mathbb{R}^6} y_{10} \]

s.t. \[
\begin{pmatrix}
y_{00} & y_{10} & y_{01} \\
y_{10} & y_{20} & y_{11} \\
y_{01} & y_{11} & y_{02}
\end{pmatrix} \succeq 0
\]

\[ 2y_{10} - y_{01} + 1 \geq 0 \]
\[ 2y_{10} + y_{01} + 1 \geq 0 \]
\[ y_{20} + y_{02} - 1 = 0 \]
\[ y_{00} = 1 \]

<table>
<thead>
<tr>
<th>bound value</th>
<th>( \rho_{1}^{\text{MOM}} )</th>
<th>( \rho_{2}^{\text{MOM}} )</th>
<th>( \bar{\rho} ) (NLP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.50</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
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</table>
Convergence of the Moment-SOS hierarchy of semidefinite programs

\[ \rho = \min_{x \in X} f(x) \]

\[ \rho_{d+1}^{\text{MOM}} \geq \rho_{d+1}^{\text{SOS}} \]

\[ \rho_{d}^{\text{MOM}} \geq \rho_{d}^{\text{SOS}} \]

\[ (d \geq \max_{j \in [0,m]} d_j) \]
1. Background on the moment hierarchy for POPs

2. The AC-OPF problem: POP formulation and scalability issue

3. Addressing large scale instances

4. Conclusion and perspective
Power grid data

PGLib’s case 14 IEEE
Notations for the AC-OPF problem

\[
\begin{align*}
\text{min} & \quad \sum_{g \in G} C_{2,g} \Re(s_g)^2 + C_{1,g} \Re(s_g) + C_{0,g} \\
\text{s.t.} & \quad \angle v_i = 0, \quad \forall i \in \mathcal{N}_r \\
& \quad S_g \leq s_g \leq \overline{S}_g, \quad \forall g \in \mathcal{G} \\
& \quad V_i \leq |v_i| \leq \overline{V}_i, \quad \forall i \in \mathcal{N}
\end{align*}
\]
Notations for the AC-OPF problem

\[
\begin{align*}
\min_{v \in \mathbb{C}^{|N|}, s \in \mathbb{C}^{|G|}, s^\ell \in \mathbb{C}^2|\mathcal{E}|} & \quad \sum_{g \in \mathcal{G}} C_{2,g} \Re(s_g)^2 + C_{1,g} \Re(s_g) + C_{0,g} \\
\text{s.t.} & \quad \angle v_i = 0, \quad \forall i \in \mathcal{N}_r \\
& \quad \underline{S}_g \leq s_g \leq \overline{S}_g, \quad \forall g \in \mathcal{G} \\
& \quad \underline{V}_i \leq |v_i| \leq \overline{V}_i, \quad \forall i \in \mathcal{N} \\
& \quad \sum_{g \in \mathcal{G}(i)} s_g - L_i - (Y_i^s)^* |v_i|^2 = \sum_{j \in \mathcal{N}(i)} s_{i,j}^\ell, \quad \forall i \in \mathcal{N} \\
& \quad s_{i,j}^\ell = (Y_{i,j} + Y_{i,j}^c)^* \frac{|v_i|^2}{|T_{i,j}|^2} - Y_{i,j}^* \frac{v_i v_j^*}{T_{i,j}}, \quad \forall (i,j) \in \mathcal{E} \\
& \quad s_{j,i}^\ell = (Y_{i,j} + Y_{j,i}^c)^* |v_j|^2 - Y_{i,j}^* \frac{v_i^* v_j}{T_{i,j}^*}, \quad \forall (i,j) \in \mathcal{E}
\end{align*}
\]
Notations for the AC-OPF problem

\[
\begin{align*}
\min_{v \in \mathbb{C}^{\left|\mathcal{N}\right|}} & \sum_{g \in \mathcal{G}} C_{2,g} \Re(s_g)^2 + C_{1,g} \Re(s_g) + C_{0,g} \\
\text{s.t.} & \angle v_i = 0, \ \forall i \in \mathcal{N}_r \\
& S_g \leq s_g \leq \overline{S}_g, \ \forall g \in \mathcal{G} \\
& V_i \leq |v_i| \leq \overline{V}_i, \ \forall i \in \mathcal{N} \\
& \sum_{g \in \mathcal{G}(i)} s_g - L_i - (Y^s_i)^* |v_i|^2 = \sum_{j \in \mathcal{N}(i)} s^\ell_{i,j}, \ \forall i \in \mathcal{N} \\
& s^\ell_{i,j} = (Y_{i,j} + Y^c_{i,j})^* \frac{|v_i|^2}{|T_{i,j}|^2} - Y^*_{i,j} \frac{v_i v_j^*}{T^*_{i,j}}, \ \forall (i,j) \in \mathcal{E} \\
& s^\ell_{j,i} = (Y_{i,j} + Y^c_{j,i})^* |v_j|^2 - Y^*_{i,j} \frac{v_i^* v_j}{T^*_{i,j}}, \ \forall (i,j) \in \mathcal{E} \\
& |s^\ell_{i,j}| \leq \overline{S}_{i,j}, \ \forall j \in \mathcal{N}(i), \ \forall i \in \mathcal{N} \\
& \Theta_{i,j} \leq \angle v_i v_j^* \leq \overline{\Theta}_{i,j}, \ \forall (i,j) \in \mathcal{E}
\end{align*}
\]
Polynomial optimization for AC-OPF

\[ v_i = a_i + \text{i}b_i \, , \, \forall i \in [1, n] \]

**Example (complex line power)**

\[ s_{i,j}^l = Z_{i,j}|v_i|^2 + Z'_{i,j}v_i v_j^* \]

\[ \iff \]

\[
\begin{align*}
\mathcal{R}(s_{i,j}^l) &= \mathcal{R}(Z_{i,j})(a_i^2 + b_i^2) + \mathcal{R}(Z'_{i,j})(a_i a_j + b_i b_j) - \mathcal{S}(Z'_{i,j})(a_j b_i - a_i b_j) \\
\mathcal{S}(s_{i,j}^l) &= \mathcal{S}(Z_{i,j})(a_i^2 + b_i^2) + \mathcal{S}(Z'_{i,j})(a_i a_j + b_i b_j) + \mathcal{R}(Z'_{i,j})(a_j b_i - a_i b_j)
\end{align*}
\]

The AC-OPF problem can be written in form (POP)!
Scalability issue

- AC-OPF IEEE case 57 (no line/angle limits) → POP

<table>
<thead>
<tr>
<th></th>
<th>(POP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>variables</td>
<td>128</td>
</tr>
<tr>
<td>eq. constraints</td>
<td>115</td>
</tr>
<tr>
<td>ineq. constraints</td>
<td>128</td>
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ρ\_MOM = 2 for PGLib’s case 57 IEEE is intractable! (with LAAS computers and current SDP solvers)
Scalability issue

- AC-OPF IEEE case 57 (no line/angle limits) → POP

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- POP → moment relaxation

<table>
<thead>
<tr>
<th></th>
<th>$d = 1$</th>
<th>$d = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>size($y$)</td>
<td>8.385</td>
<td>12.082.785</td>
</tr>
</tbody>
</table>

$\rho_2^{MOM}$ for PGLib’s case 57 IEEE is intractable!
(with LAAS computers and current SDP solvers)
1. Background on the moment hierarchy for POPs

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3. Addressing large scale instances?

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3. Addressing large scale instances?

Correlative sparsity

Minimal sparsity
Exploiting sparsity for POPs

\[
\min_{x \in \mathbb{R}^3} x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3
\]

Exploit absence of \(x_1 x_3\) product?
Exploiting sparsity for POPs

\[
\min_{x \in \mathbb{R}^3} x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3
\]

Exploit absence of \(x_1 x_3\) product? Set \(\mathcal{I}_1 = \{1, 2\}, \mathcal{I}_2 = \{2, 3\}\)

\[
M_1(y) = \begin{pmatrix}
y_{000} & y_{100} & y_{010} & y_{001} \\
y_{100} & y_{200} & y_{110} & y_{101} \\
y_{010} & y_{110} & y_{020} & y_{011} \\
y_{001} & y_{101} & y_{011} & y_{002}
\end{pmatrix} \succeq 0 \text{ vs } \begin{cases} M_1(y|\mathcal{I}_1) \succeq 0 \\
M_1(y|\mathcal{I}_2) \succeq 0 \end{cases}
\]
Exploiting sparsity for POPs

\[
\min_{x \in \mathbb{R}^3} x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3
\]

Exploit absence of \(x_1 x_3\) product? Set \(I_1 = \{1, 2\}, I_2 = \{2, 3\}\)

\[
M_1(y) = \begin{pmatrix}
  y_{000} & y_{100} & y_{010} & y_{001} \\
  y_{100} & y_{200} & y_{110} & y_{101} \\
  y_{010} & y_{110} & y_{020} & y_{011} \\
  y_{001} & y_{101} & y_{011} & y_{002}
\end{pmatrix} \succeq 0 \text{ vs } \begin{cases}
M_1(y|I_1) \succeq 0 \\
M_1(y|I_2) \succeq 0
\end{cases}
\]

Reduce moment variables (MOM\(_d\)) / matrices size (SOS\(_d\))
A sparse moment hierarchy

\[
\rho_{d}^{\text{CS-MOM}} = \inf_{y} \sum_{\alpha \in \text{supp}(f)} f_{\alpha} y_{\alpha} \quad \text{(CS-MOM}_{d})
\]

s.t. \( M_{d}(y|I_{k}) \succeq 0 \), \( \forall k \in [1, p] \)

\( M_{d-d_{j}}(g_{j}y|I_{k}) \succeq 0 \), \( \forall j \in [1, m] \), \( \forall k \in [1, p] \)

\( y_{0, \ldots, 0} = 1 \)
A sparse moment hierarchy

\[
\rho_{d}^{\text{CS-MOM}} = \inf_{y} \sum_{\alpha \in \text{supp}(f)} f_{\alpha} y_{\alpha} \tag{\text{CS-MOM}_{d}}
\]

s.t. \[ M_{d}(y|I_{k}) \succeq 0, \ \forall k \in [1, p] \]
\[ M_{d-d_{j}}(g_{j}y|I_{k}) \succeq 0, \ \forall j \in [1, m], \ \forall k \in [1, p] \]
\[ y_{0,\ldots,0} = 1 \]

**Theorem (Lasserre [2006])**

*If the set \( \mathcal{X} \) is compact and satisfies an Archimedeaness property, and if the variable set \( \mathcal{I} = \{I_{k}\}_{k \in [1,p]} \) satisfies the running intersection property (RIP), then the monotonous non-decreasing sequence of values \( \{\rho_{d}^{\text{CS-MOM}}\}_{d \in \mathbb{N}} \) of \( \text{CS-MOM}_{d} \) converges to the value \( \rho \) of \( \text{POP} \).*

NB: the maximum cliques of a chordal graph satisfy the RIP
Application to AC-OPF

IEEE case 57 after chordal extension + cliques:

\[
\begin{align*}
|\mathcal{I}| &= 52 \\
\max_{k \in [1, p]} |\mathcal{I}_k| &= 26
\end{align*}
\]

- POP → sparse moment relaxation

\[
\begin{array}{c|c|c}
 & d = 1 & d = 2 \\
\hline
\text{size}(y) & 1.950 & 122.286 \\
\end{array}
\]

- numerical result (IEEE case 57 perturbed):

<table>
<thead>
<tr>
<th></th>
<th>value</th>
<th>gap to $\bar{\rho}$ (%)</th>
<th>time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\rho}$</td>
<td>2433.89</td>
<td>-</td>
<td>4.18</td>
</tr>
<tr>
<td>$\rho^\text{CS-MOM}_2$</td>
<td>2433.89</td>
<td>0.00</td>
<td>19,666.82</td>
</tr>
<tr>
<td>$\rho^\text{CS-MOM}_1$</td>
<td>2359.58</td>
<td>3.05</td>
<td>0.75</td>
</tr>
</tbody>
</table>
3. Addressing large scale instances?

   Correlative sparsity

   Minimal sparsity
Sparsity sets and scalability

- Interior point SDP solvers scale roughly in $O(N^3)$ with $N = \binom{m+d}{d}$ and $m = \max_{k \in [1,p]} |\mathcal{I}_k|$

- It is difficult to control the cardinalities of the sets $\{\mathcal{I}_k\}_{k \in [1,p]}$ obtained by chordal extension + cliques

We introduce **minimal sparsity** designed to reduce the cardinalities of the sets $\{\mathcal{I}_k\}_{k \in [1,p]}$ in AC-OPF
Minimal sparsity based on power flow equations

\[ \{x_n\}_{n \in \mathcal{I}_{\#i}^m} = \{\mathcal{R}(v_i), \mathcal{I}(v_i)\} \bigcup_{j \in \mathcal{N}(i)} \{\mathcal{R}(v_j), \mathcal{I}(v_j)\} \bigcup_{g \in \mathcal{G}(i)} \{\mathcal{R}(s_g), \mathcal{I}(s_g)\} \]
More but smaller sparsity sets

PGLib’s case 57 IEEE

![Graph showing clique-based sparsity and minimal sparsity]
Second-order moment relaxations via minimal sparsity

- PGLib’s case 57 IEEE

<table>
<thead>
<tr>
<th></th>
<th>size(y)</th>
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<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_2$</td>
<td>12,082,785</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$\rho_2^{\text{CS-MOM}}$</td>
<td>122,286</td>
<td>19,666</td>
<td>2433.89</td>
</tr>
<tr>
<td>$\rho_2^{\text{MS-MOM}}$</td>
<td>23,526</td>
<td>45</td>
<td>2433.89</td>
</tr>
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</table>
Second-order moment relaxations via minimal sparsity

- PGLib’s case 57 IEEE

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- Large scale instances?

<table>
<thead>
<tr>
<th>cases</th>
<th>gap (%)</th>
<th>time (s)</th>
</tr>
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<tbody>
<tr>
<td>2868 RTE SAD</td>
<td>0.39</td>
<td>6,981</td>
</tr>
<tr>
<td>6468 RTE TYP</td>
<td>0.27</td>
<td>12,723</td>
</tr>
<tr>
<td>6470 RTE TYP</td>
<td>0.74</td>
<td>15,662</td>
</tr>
</tbody>
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2. The AC-OPF problem: POP formulation and scalability issue
3. Addressing large scale instances?
4. Conclusion and perspective
Further improvements for very large scale problems

AC-OPF formulates as a POP but...
typical instances in France have over 6000 nodes!

- Sparsity can help to address very large problems
- Minimal sparsity looks promising
to compute second-order relaxations of large instances
- We obtain very large SDPs whose numerical stability
  needs to be improved (future work)